APPENDIX A

Fourier Series and Integrals (fundamental principles)

The following fundamentals and elementary facts are standard mathematical knowledge today, and can be found in a great number of text books in analysis. As a general reference, we mention [**DM**, 1972].

1. Fourier Series

We use the notation $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and define the function spaces

$$\begin{aligned} C^0(S^1) &:= \left\{ \begin{array}{l} \text{the Banach space of continuous } \mathbb{C}\text{-valued functions on } S^1 \text{ with} \\ \text{the norm } \|f\|_\infty := \sup\left\{|f(z)| : z \in S^1\right\}, \end{array} \right. \\ L^1(S^1) &:= \left\{ \begin{array}{l} \text{the Banach space of } \mathbb{C}\text{-valued, integrable functions on } S^1 \text{ with} \\ \text{the norm } \|f\|_1 := \int_{S^1} |f|, \end{array} \right. \\ L^2(S^1) &:= \left\{ \begin{array}{l} \text{the Hilbert space of square-integrable } \mathbb{C}\text{-valued functions on } S^1 \\ \text{with inner product } \langle f, g\rangle := \int_{S^1} f\bar{g} \text{ and norm } \|f\|_2 := \sqrt{\langle f, f\rangle}. \end{array} \right. \end{aligned}$$

Warning: Functions in $L^1(S^1)$ or $L^2(S^1)$ are identified if they agree outside a set of measure zero. In particular, f is identified with the zero function, if f is zero "almost everywhere"; i.e., f is nonzero only on a set of measure zero. In this way, we have $\|f\| = 0$ precisely when $f = 0$. Thus, strictly speaking, the elements of $L^1(S^1)$ or $L^2(S^1)$ are not functions, but rather equivalence classes of functions. While this is true, in practice it is much simpler and generally harmless to disregard this fine distinction, and we will do this in what follows. Moreover, it is convenient to regard $L^1(S^1)$ as $L^1([0,1])$, and $L^2(S^1)$ as $L^2([0,1])$, and we will do so often without comment. Then $\|f\|_1 := \int_0^1 |f(x)|\, dx$ and $\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}\, dx$. Perhaps $f(0) \neq f(1)$, but this does not matter in L^1 or L^2 since $\{0,1\}$ is of measure 0. However, $C^0(S^1)$ and $C^0([0,1])$ are not naturally identified.

Exercise A.1. Show that $L^2(S^1)$ with $\langle\cdot,\cdot\rangle$ is indeed a Hilbert space. We need to show:

a) $\langle\cdot,\cdot\rangle : L^2(S^1) \times L^2(S^1) \to \mathbb{C}$ is well defined. [Hint: The pointwise estimate

$$2|f(z)\overline{g(z)}| = 2\,|f(z)|\,|g(z)| \leq |f(z)|^2 + |g(z)|^2$$

shows that $f\bar{g} \in L^1(S^1)$ for $f, g \in L^2(S^1)$.]

b) $\langle\cdot,\cdot\rangle$ is sesquilinear, namely $\langle f, g\rangle$ is $\mathbb{C}$-linear in f, $\langle f, g\rangle = \overline{\langle g, f\rangle}$ (i.e., $\langle f, g\rangle$ is conjugate linear in g). Also $\langle\cdot,\cdot\rangle$ is positive; i.e., $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0$ only for $f = 0$. (All of this is trivial.)

c) $L^2(S^1)$ is a complex vector space. [Hint: For closure under addition, prove Hermann Minkowski's inequality $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$ (the Triangle Inequality).]

d) $L^2(S^1)$ is complete. In order to prove that a Cauchy sequence $\{f_n\}$ in $L^2(S^1)$

(i.e., $\|f_n - f_m\|_2 \to 0$ as $n, m \to \infty$) possesses a limit $f \in L^2(S^1)$ (i.e., $\|f_n - f\|_2 \to 0$ as $n \to \infty$), one applies the fundamental convergence theorems which distinguish the Lebesgue integral from the Riemann integral. The rather technical proof can be found in [**DM**, p.16-20].

e) $L^2(S^1)$ is separable. [Hint: Show that the family of piecewise constant functions, having rational real and imaginary parts and with jumps at finitely many rational points, is dense in $L^2(S^1)$.]

Approximation: With the help of the smoothing functions of the kind

$$g(x) := \begin{cases} c \exp\left((x-a)^{-1} (x-b)^{-1} \right) & \text{for } a < x < b \\ 0 & \text{for } x \le a \text{ or } x \ge b \end{cases}$$

it follows that $C^\infty(S^1)$ is dense in $L^2(S^1)$.

Convolution: In $L^1(S^1)$, there is a commutative and associative product (known as "convolution") given by

$$(f * g)(x) := \int_0^1 f(x-y) g(y)\, dy, \ x \in [0,1],$$

where we assume that f is extended periodically of period 1 so that $f(x-y)$ makes sense when $x - y \notin [0,1]$. This makes $L^1(S^1)$ an algebra (without identity). By applying the theorem of Guido Fubini on iterated integrals, we obtain

$$\|f * g\|_1 \le \|f\|_1 \ \|g\|_1 .$$

Moreover, one can show that, relative to $*$, $L^2(S^1)$ is an ideal in $L^1(S^1)$ hence, $f * g$ is in $L^2(S)$, whenever one of the factors lies in $L^2(S)$. See [**DM**, p.41].

Orthonormal systems: The family $\{z^n : n \in \mathbb{Z}\}$, where $z^n : S^1 \to \mathbb{C}$ is the function that assigns to each $z \in S^1$ the value z^n, is a complete orthonormal system in $L^2(S^1)$. Regarding $L^2(S^1)$ as $L^2([0,1])$, the corresponding functions have the form $e^{2\pi i n x}$.

Fourier series: This orthonormal system is complete; i.e., its linear span is dense in $L^2(S^1)$. Because of this, each function $f \in L^2(S^1)$ can be expanded in a Fourier series

$$f = \sum\nolimits_{n=-\infty}^{\infty} \widehat{f}(n) z^n \ \ \text{(i.e., } \lim_{k \to \infty} \| f - \sum\nolimits_{n=-k}^{k} \widehat{f}(n) z^n \|_2 = 0).$$

with the Fourier coefficients

$$\widehat{f}(n) := \langle f, z^n \rangle = \int_0^1 f(x) e^{-2\pi i n x}\, dx. \tag{1.1}$$

Note that f equals its infinite Fourier series, in the sense that the partial sums $\sum_{|n| \le k} \widehat{f}(n) z^n$ converge to f in the $L^2(S^1)$-norm as $k \to \infty$, but not necessarily pointwise. The function

$$f \mapsto \left\{ \widehat{f}(n) \right\}_{n=-\infty}^{\infty} = \ldots, \widehat{f}(-1), \widehat{f}(0), \widehat{f}(1), \ldots$$

is an isomorphism from $L^2(S^1)$ to the space $L^2(\mathbb{Z})$ of absolute square-summable sequences of complex numbers. The isomorphism is an isometry, namely

$$\|f\|_2^2 = \sum\nolimits_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \ \ \text{(Plancherel's Identity)}.$$

Details are in [**DM**]. The Fourier coefficients are also defined for $f \in L^1(S^1)$, and by Fubini's Theorem, it then follows [**DM**, p.42] that

$$(f * g)\widehat{\ }(n) = \widehat{f}(n)\widehat{g}(n).$$

Incidentally, the algebra A of those sequences which appear as the Fourier coefficients of integrable functions has been barely investigated: "The best information available to date indicates that A has no decent description at all." [**DM**, p.43].

On the other hand, when the product fg (in the sense of pointwise multiplication) is integrable (e.g., when $f, g \in L^2(S)$, by Exercise A.1a), then the Fourier coefficients satisfy

$$(fg)\widehat{\ }(n) = (\widehat{f} * \widehat{g})(n) := \sum\nolimits_{k=-\infty}^{\infty} \widehat{f}\,(n-k)\,\widehat{g}(k).$$

For the proof, we do not need Fubini's Theorem as above, but rather we insert the Fourier series of g in the formula for $(fg)\widehat{\ }(n)$, and then use the usual limit theorems for the Lebesgue integral, to interchange the integral and sum.

2. The Fourier Integral

One can proceed from the standard representation of functions on a circle as functions of period 1 on the real line, to the more general case of period T, and then let T go to ∞. This leads to the concept of the Fourier transform. Let $L^1(\mathbb{R})$ denote the Banach space of integrable functions with

$$\|f\|_1 := \int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

In [**DM**, p.86f.], the Fourier transform of $f \in L^1(\mathbb{R})$ is defined as

$$\widehat{f}_{DM}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}dx, \tag{2.1}$$

which is a natural extension of (1.1), namely $\widehat{f}(n) := \int_0^1 f(x)e^{-i2\pi nx}dx$. In the previous edition [**BlBo85**, p.82] of this book, the Fourier transform was given as

$$\widehat{f}_B(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \widehat{f}_{DM}(\xi/2\pi).$$

In [**Rud**, p.167] (and [**BlCs**, p.423]), we find the definition

$$\widehat{f}_R(\xi) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \frac{1}{\sqrt{2\pi}}\widehat{f}_B(\xi) = \frac{1}{\sqrt{2\pi}}\widehat{f}_{DM}(\xi/2\pi).$$

In Remark A.3 (below, p.625), there are cogent reasons for adopting any one of the above definitions. Since our emphasis in this book is on the application of Fourier transforms to differential equations and we wish the Fourier transform to be an L^2 isometry, we use $\widehat{f}_R$ as Remark A.3 suggests. Thus, we define the **Fourier transform** of $f \in L^1(\mathbb{R})$ via

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx. \tag{2.2}$$

Since it is an irritating annoyance to have to include the factor $1/\sqrt{2\pi}$, we adopt the notation

$$d'x := dx/\sqrt{2\pi}, \text{ so that } \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x} d'x.$$

We consider the Fourier transformation on the spaces $C_{\downarrow}^{\infty}(\mathbb{R})$, $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Here, $C_{\downarrow}^{\infty}(\mathbb{R})$ is the space of "rapidly decreasing" C^{∞} functions on $\mathbb{R}$ (with complex values). "Rapidly decreasing" means that these functions and all their derivatives tend to 0 at infinity, even when they are multiplied by arbitrary polynomials. $L^1(\mathbb{R})$ is an algebra (without identity) under convolution

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)dy,$$

and we have [**DM**, p.87f.]:

$$\|f * g\|_1 \leq \|f\|_1 \ \|g\|_1 .$$

$L^2(\mathbb{R})$ is the Hilbert space (proved as in Exercise A.1) of square integrable functions with

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \text{ and } \|f\|_2 := \sqrt{\langle f, f\rangle}$$

By the argument of Exercise A.1a, it follows that $C_{\downarrow}^{\infty}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, as well as in $L^2(\mathbb{R})$. Naturally one cannot expect, as with functions on the (compact) circle S^1, that $C^{\infty}(\mathbb{R})$ or $C^0(\mathbb{R})$ will be contained in the Lebesgue spaces.

A further complication arises since we no longer have $L^2 \subseteq L^1$, or even $L^1 \subseteq L^2$. For example, if $f(x) = 1$ for $|x| < 1$ and $f(x) = 0$ for $|x| \geq 1$, then

$$f(x)\,|x|^{-2/3} \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R}) \quad\text{and}\quad (1 - f(x))\,|x|^{-2/3} \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R}).$$

Exercise A.2. The following statements are easy to prove:
(a) For each $f \in L^1(\mathbb{R})$ and each $\xi \in \mathbb{R}$, $\widehat{f}(\xi)$ is well defined.
(b) If $f \in C^1(\mathbb{R})$, $f' \in L^1(\mathbb{R})$ and $ixf(x)$ stands for the function $x \mapsto ixf(x)$ which is assumed to be in $L^1(\mathbb{R})$, then we have

$$(f')\widehat{\ }(\xi) = i\xi\widehat{f}(\xi) \text{ and} \tag{2.3}$$

$$(\widehat{f})'(\xi) = -(ixf(x))\widehat{\ }(\xi). \tag{2.4}$$

The proof of each is via integration by parts. Using the following notation for the various operations above

$$(Df)(x) := \frac{1}{i}f'(x), \ (Mf)(x) := xf(x), \text{ and } Ff := \widehat{f},$$

we have

$$FDf = MFf \text{ and } DFf = -FMf.$$

More generally, by induction, we have (for $f \in C_{\downarrow}^{\infty}(\mathbb{R})$ and $p, q \in \mathbb{N}$)

$$M^p D^q F f = (-1)^q \, F M^q D^p f.$$

This formula is of fundamental importance for the treatment of differential operators (with constant coefficients) which are converted into simple multipliers. See Chapter 7 on pseudo-differential operators. From the topological viewpoint these formulas are most remarkable, because they express a deep duality between local and global properties: Thus (2.3) relates the smoothness of f with the rate of decay (asymptotic behavior) of $\widehat{f}$, and (2.4) relates the smoothness of $\widehat{f}$ with the decay of f. In fact, $\widehat{f}$ is differentiable (a local property) when f decreases so fast that the Fourier integral of $-ixf(x)$ converges. This local–global duality is also a feature of the index formula for elliptic operators, and we will deal with it further in that context.

(c) The **Fourier Inversion Formula**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} d\xi$$

holds for $f \in C^\infty_\downarrow(\mathbb{R})$. In direct analogy with the role of Fourier coefficients in Fourier series, $\widehat{f}(\xi)$ is the density of the frequency ξ in the harmonic decomposition of f.

[Hint: Prove the formula first for functions with compact support (i.e., vanishing outside a compact subset of $\mathbb{R}$). In this case there are no difficulties with the limit process which reduces to functions of period T and then let T go to infinity. Use smoothing functions as following Exercise A.1e in order to approximate rapidly decreasing C^∞ functions by functions of compact support. The $L^1(\mathbb{R})$ estimates needed next are somewhat tricky, but can be looked up in [**DM**, p.89f]. A shorter direct proof can be found in [**Hö63**, 1963, p.18f].]

(d) As a corollary to the proof of (c), one obtains the **Plancherel formula**

$$\|\widehat{f}\|_2 = \|f\|_2$$

and that

$$F : C^\infty_\downarrow(\mathbb{R}) \to C^\infty_\downarrow(\mathbb{R})$$

is linear and bijective, where F again denotes the Fourier transformation. By the Fourier Inversion Formula, we obtain the inverse transformation

$$\left(F^{-1}f\right)(x) = (Ff)(-x).$$

(e) Extend F from $C^\infty_\downarrow(\mathbb{R})$ to $L^2(\mathbb{R})$! [Hint: Approximate $f \in L^2(\mathbb{R})$ in the $L^2(\mathbb{R})$-norm by a sequence $\{f_n\}$ with $f_n \in C^\infty_\downarrow(\mathbb{R})$. Using the additivity of F and the Plancherel Identity, show that $\{\widehat{f_n}\}$ is a Cauchy sequence in $L^2(\mathbb{R})$, whence $\widehat{f} := \lim \widehat{f_n}$ defines an element of the Hilbert space $L^2(\mathbb{R})$. Finally check that $\widehat{f}$ indeed depends only on f and not on the choice of the sequence. In this way, one obtains an isomorphism from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, which we denote by F again.]

(f) The spaces $C^\infty_\downarrow(\mathbb{R})$ and $L^2(\mathbb{R})$ share the property that they are mapped into themselves by F. This is not true for $L^1(\mathbb{R})$. Still, one can easily show [**DM**, p.102] that for $f \in L^1(\mathbb{R})$,

$$\begin{aligned}
&(i)\ \widehat{f} \in C^0(\mathbb{R})\\
&(ii)\ \lim\nolimits_{|\xi|\to\infty} \widehat{f}(\xi) = 0\\
&(iii)\ (f * g)\widehat{\ }(\xi) = \sqrt{2\pi}\,\widehat{f}(\xi)\,\widehat{g}(\xi)\,.
\end{aligned}$$

Remark A.3. Here we consider the merits of the three definitions

$$\begin{aligned}&\widehat{f}_{DM}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}dx,\\ &\widehat{f}_{B}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \widehat{f}_{DM}(\xi/2\pi) \text{ and}\\ &\widehat{f}_{R}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i\xi x}d'x := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \frac{1}{\sqrt{2\pi}}\widehat{f}_{DM}(\xi/2\pi),\end{aligned}$$

found in [**DM**, p.87], [**BlBo85**, p.82], and [**Rud**, p.167]. The main advantage of $\widehat{f}_B$ is that there is no 2π or $\sqrt{2\pi}$. However, in terms of $\widehat{f}_B$, the Plancherel formula becomes $\|\widehat{f}_B\|_2 = \sqrt{2\pi}\,\|f\|_2$ so that $f \mapsto \widehat{f}_B$ is not an isometry, which is a drawback. We do have $\|\widehat{f}_{DM}\|_2 = \|f\|_2$. Moreover, $\widehat{f}_{DM}$ is good for expressing the remarkable **Poisson summation formula**

$$\sum\nolimits_{k=-\infty}^{\infty} f(kL) = \frac{1}{L}\sum\nolimits_{k=-\infty}^{\infty} \widehat{f}_{DM}\left(k/L\right), \tag{2.5}$$

which holds for $f \in C_{\downarrow}^{\infty}(\mathbb{R})$ and any $L > 0$. This relates the sum of f over the lattice $\{kL : k \in \mathbb{Z}\}$ to the sum of $\widehat{f}$ over the reciprocal lattice $\{k/L : k \in \mathbb{Z}\}$. Using $\widehat{f}_R$ or $\widehat{f}_B$, this becomes

$$\sum\nolimits_{k=-\infty}^{\infty} f(kL) = \frac{\sqrt{2\pi}}{L}\sum\nolimits_{k=-\infty}^{\infty} \widehat{f}_{R}\left(2\pi k/L\right) = \frac{1}{L}\sum\nolimits_{k=-\infty}^{\infty} \widehat{f}_{B}\left(2\pi k/L\right)$$

which is less esthetic and harder to recall. The convolution theorem gives $(f * g)\widehat{\ }_B = \widehat{f}_B\widehat{g}_B$ and $(f * g)\widehat{\ }_{DM} = \widehat{f}_{DM}\widehat{g}_{DM}$, both of which look better than $(f * g)\widehat{\ }_R = \sqrt{2\pi}\widehat{f}_R\widehat{g}_R$. So far, $\widehat{f}_{DM}$ seems to be the best choice. However,

$$(f')\widehat{\ }_{DM}\ (\xi) = 2\pi i\xi\widehat{f}_{DM}(\xi),$$

and the excess baggage of the 2π makes $\widehat{f}_{DM}$ a bit cumbersome for applications to differential equations. Thus, we have adopted $\widehat{f}_R$, but not passionately and not exclusively; indeed in almost all of Chapter 4, we use $\widehat{f}_B$ since the factor $1/\sqrt{2\pi}$ only serves as a needless distraction.

3. Higher Dimensional Fourier Integrals

By the theorem of Guido Fubini, the closed linear span of the n-fold products

$$f_1(x_1)\cdots f_n(x_n)$$

of functions in $L^2(\mathbb{R})$ is in $L^2(\mathbb{R}^n)$ (Prove!). Thus, the preceding results carry over directly to the case of several variables:

Exercise A.4. As above, define the spaces $C_{\downarrow}^{\infty}(\mathbb{R}^n)$, $L^1(\mathbb{R}^n)$, and $L^2(\mathbb{R}^n)$, and investigate the Fourier transform

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi\rangle}d'x := (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi\rangle}dx,$$

where $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$, $\langle x, \xi\rangle = x_1\xi_1 + ... + x_n\xi_n$, and $d'x = (2\pi)^{-n/2}dx_1\cdots dx_n$. Show:

(a) The Fourier Inversion Formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{i\langle x,\xi\rangle}d'\xi, \text{ where } f \in C_{\downarrow}^{\infty}(\mathbb{R}^n).$$

(b) For multi-indices p, q, with $|q| = q_1 + \cdots + q_n$ and $Ff := \widehat{f}$,

$$M^p D^q F = (-1)^{|q|} F D^p M^q, \text{ where } f \in C_{\downarrow}^{\infty}(\mathbb{R}^n) \text{ and}$$

$$D^q := (-1)^{|q|} \frac{\partial^{|q|}}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}.$$

(c) $f \in L^1(\mathbb{R}^n) \Rightarrow \widehat{f} \in C^0(\mathbb{R}^n)$.

(d) Parseval's equality $\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi$, for $f, g \in L^2(\mathbb{R}^n)$.

(e) $\widehat{fg} = (2\pi)^{n/2} \left(\widehat{f} * \widehat{g}\right)$ and $\widehat{f}\widehat{g} = (2\pi)^{-n/2} (f * g)\widehat{\ }$, if $f, g, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$.

Details are in [**DM**, p.132f] or [**Hö63**, 1963, p.17-19].

APPENDIX B

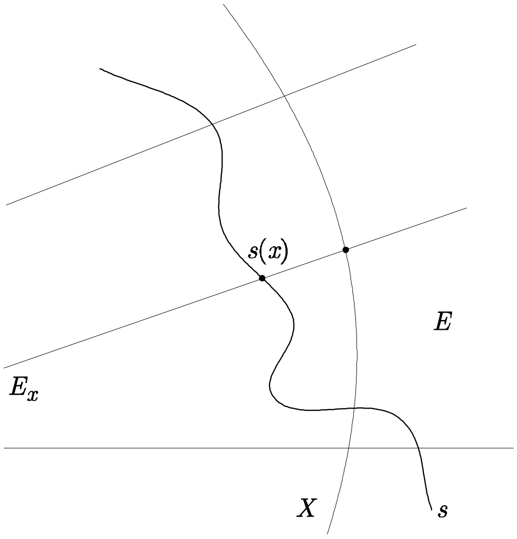
Vector Bundles

Let X be a topological space. A **family of vector spaces** over is a topological space E together with

(i) a continuous surjective map $p : E \to X$ and

(ii) a vector space structure of finite dimension in each $E_x = p^{-1}(x)$, which carries the topology induced by E.

By "vector spaces", we mean complex vector spaces, unless explicitly indicated otherwise. The mapping p is called the **projection**, E is called the **total space** of the family, X is the **parameter space** or **base space** of the family for $x \in X$, E_x is the **fiber** over x. A section of a family $p : E \to X$ is a continuous map $s : X \to E$ such that $(p \circ s)(x) = x$ for all $x \in X$.



A **homomorphism** from one family $p : E \to X$ to another $q : F \to X$ is a continuous map $\phi : E \to F$ such that

(i) $q \circ \phi = p$ and

(ii) $\phi_x : E_x \to F_x$ is a linear map for each $x \in X$. We write $\phi \in \mathrm{Hom}(E, F)$.

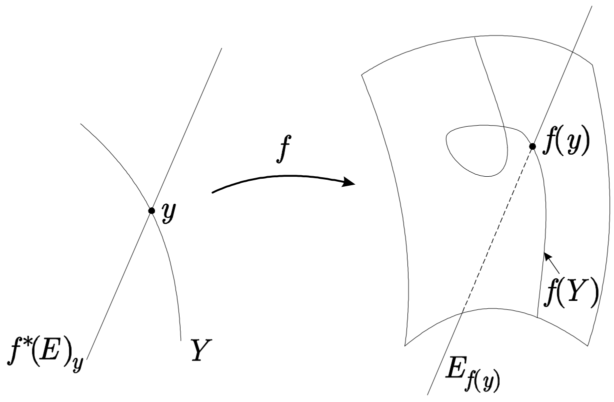
We say that such a ϕ is an isomorphism when ϕ is bijective and ϕ^{-1} is continuous. E and F are called isomorphic when there is an isomorphism between them. We write $\phi \in \mathrm{Iso}(E, F)$ and $E \cong F$.

Exercise B.1. a) Let V be a finite-dimensional vector space; e.g., $V = \mathbb{C}^N$. Show that a family of vector spaces over X is obtained by taking $E := X \times V$ with $p : E \to X$ being the projection on the first factor. This is the product family V_X

with fiber V.
b) If F is a family which is isomorphic to a product family, then one calls F **trivial**. Show that a trivial family of finite dimensional (real) vector spaces is obtained, if

$$\begin{aligned}&E := \big\{(x, y, -\lambda y, \lambda x) : x, y, \lambda \in \mathbb{R} \text{ and } x^2 + y^2 = 1\big\} \text{ and}\\ &p(x, y, *, *) := (x, y).\end{aligned}$$

In general, prove that a bundle F is isomorphic to a product family V_X, if and only if one can find N sections $s_i : X \to P$ such that $s_1(x), ..., s_N(x)$ forms a basis for F_x for each $x \in X$; here $N = \dim V$.
c) Let Y be a subset of X and E a family of vector spaces over X with projection p. Show that $p^{-1}(Y) \to Y$ is a family over Y. We call this the **restriction** of E to Y, and write $E|Y$ for this family.
d) More generally: Let Y be an arbitrary topological space and $f : Y \to X$ a continuous map. As follows, define the **induced family** $f^*(p) : f^*(E) \to Y$: Take $f^*(E)$ to be the subspace of $Y \times E$ consisting of points (y, e) with $f(y) = p(e)$; the projection and the vector space structure on the fibers are self-evident. Show: For each further map $g : Z \to Y$, there is a natural isomorphism $(fg)^*(E) \xrightarrow{\cong} g^*f^*(E)$ which one obtains by mapping each point of the form (z, e) with $z \in Z$ and $e \in E$ to the point $(z, g(z), e)$. If $f : Y \to X$ is the inclusion, then there is an isomorphism $E|Y \xrightarrow{\cong} f^*(F)$ given by mapping $e \in E|Y$ to the point $(p(e), e)$.



A family of vector spaces is called **locally trivial**, if each $x \in X$ possesses a neighborhood U such that $E|U$ is trivial. A locally trivial family is called a **vector bundle**; trivial families are called **trivial bundles**. If $f : Y \to X$ and E is a vector bundle over clearly $f^*(E)$ is a vector bundle over Y which we call the **induced bundle**.

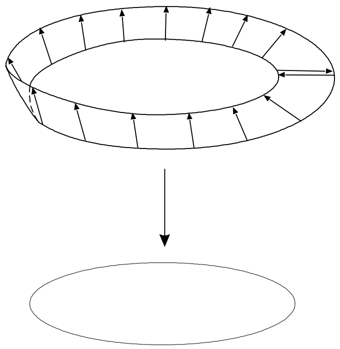
Note: If E is a vector bundle over X, then $\dim(E)$ is a locally constant function on X, and hence is constant on each connected component of X. If $\dim(E_x)$ is constant on all of X, then one says that E has **dimension** equal to the common fiber dimension $\dim(E_x)$. If X is a manifold, then the real dimension of E (regarded as a topological space) equals $\dim(X) + 2\dim(E_x)$. Vector bundles of fiber dimension 1 are also called **line bundles**.

Since a vector bundle is locally trivial, each section can be written locally as a vector-valued function on the base space. For a vector bundle E, we denote the space of sections of E by $C^0(E)$; $C^0(E)$ is a vector space in a natural way via pointwise addition, etc.

Exercise B.2. a) Let V be a (complex) vector space and $\mathbb{P}V$ its associated "projective space" of all one-dimensional linear subspaces of V. We can write $\mathbb{P}V = (V \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation $v \sim w \Leftrightarrow \lambda v = w$ for some $\lambda \in \mathbb{C}$. We define $H_V \subseteq \mathbb{P}V \times V$ as the set all (x, v) such that $x \in \mathbb{P}V$, $v \in V$, and v belongs to the complex line x. Show that H is a vector bundle in a natural way. (The construction goes back to Heinz Hopf.)
b) Go through the corresponding construction in the (more intuitive) category of real vector bundles when V is real (e.g., $\mathbb{R}^n$), and show that H_V is a real subbundle of $\mathbb{P}V \times V$ of fiber dimension 1, and that H is nontrivial if $\dim V \geq 2$.
c) For the moment, we remain in the real category and consider the following family (parametrized by $\theta \in [0, 2\pi]$) of integro-differential equations for C^∞ functions f on the unit interval which satisfy the boundary condition $f(0) = f(1)$:

$$\begin{aligned} \cos\theta\ f(x) + \sin\theta\,\frac{df}{dx} &= \cos\theta \int_0^1 f(x)\,dx,\ \theta \in [0,\pi]\,, \\ \cos\theta\ f(x) + \sin\theta\ Lf(x)) &= \cos\theta \int_0^1 f(x)\,dx + \sin\theta \int_0^1 Lf(x)\,dx,\ \theta \in [\pi, 2\pi]\,. \end{aligned}$$

Here, L is an operator with $L^2 = -\operatorname{Id}$. Show that the solutions of the family of equations forms a real vector bundle over the circle $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ that is nontrivial and isomorphic to the bundle $H_{\mathbb{R}^2}$. (Actually, every real line bundle over S^1 is either trivial or isomorphic to $H_{\mathbb{R}^2}$ see also [**BJ**, 3.23.9].)



[Hint for b): In contrast to the complex numbers, -1 cannot be deformed into 1 without going through 0. Thus, a real bundle is nontrivial, if it remains connected after the zero section is removed.
For c): First show that for $0 \leq \theta \leq \pi$, the solutions are the constant functions $c\mathbf{1}$, and for $\pi \leq \theta \leq 2\pi$ the solutions are the functions $c \cos\theta\ \mathbf{1} - \sin\theta\ L\,[\mathbf{1}]\,,\ c \in \mathbb{R}$. With the topology of $S \times C^0(I)$ or of $S^1 \times \mathbb{R}$ (since every solution can be written in the form $c_1 + c_2 L(\mathbf{1})$), construct a family of 1-dimensional (real) vector spaces over S^1 and show the local triviality. For this use the initial value map of $f \mapsto f(0)$. Note that this function can vanish for $f \neq 0$, namely if $\cos\theta_0 = \sin\theta_0\ L\,(\mathbf{1})\,(0)$. How can one proceed in a neighborhood of θ_0? Distinguish the cases where $L\,(\mathbf{1})\,(0)$ positive, negative or zero. Incidentally, how does one obtain an L with $L^2 = -\operatorname{Id}$? Start with the Fourier series $f(x) = a_0 + \sum_{\nu=1}^{\infty} (a_\nu \sin \nu x + b_\nu \cos \nu x)$, and replace a_ν by $b_{\nu+1}$ and b_ν by $-a_{\nu-1}$; see also [**Si70**].]

Remark B.3. a) and b) describe the origin of the bundle concept in analytic and projective geometry. Part c) is characteristic for many function analytic situations with "jumps" where the passage from one side to another (from one solution curve to another of the same equation) cannot be understood within the given space but requires an extension of the system (e.g., by parametrization). A basic model for such a process is present in the geometry of number fields (see Hint for b)). Many classical results of analysis – especially concerning the dependence of the solutions of a functional equation on the variation of its coefficients and on the zeros and poles of its solutions – can be aptly formulated in the language of vector bundles. Conversely, the theorem of Grothendieck, for example, that every holomorphic vector bundle E on the Riemannian number sphere $S^2 = \mathbb{P}(\mathbb{C}^2)$ can be represented as a Whitney sum $E_1 \oplus \cdots \oplus E_n$ of line bundles (Am. J. Math. **79** (1957), 121-138) was known to analysts the beginning of the century: See G. Birkhoff, Math. Ann. **54** (1913), 122-139, where Grothendieck's theorem appears as a theorem about matrices of analytic functions. Birkhoff was led to this theorem through his investigation of the singular points of ordinary differential equations; further see D. Hilbert, Gött. Nachr. (1905), 307-358, who gave a proof of Grothendieck's theorem for $N = 2$ in his "Fundamentals of a General Theory of Integral Equations" in connection with the "Riemannian Problem" (Contributed by M. Schneider).

Exercise B.4. Show that the usual operations for vector spaces in linear algebra also make sense for vector bundles. In particular, for vector bundles E and F over the same base, investigate the **direct sum** $E \oplus F$, the **tensor product** $E \otimes F$, the **homomorphism bundle** $L(E, F)$, and the **dual bundle** $E^* := L(E, \mathbb{C}_X)$. Also, carry over the concepts of subspace and quotient space from linear algebra to the corresponding concepts of a subbundle F of E and a quotient bundle E/F.
[Hint: Make use of the fact that the corresponding operations in the "structure group" $GL(N, \mathbb{C})$ are continuous! Example: set $E^* := L(E, \mathbb{C}_X)$ and introduce a topology on $E^*|U$ which makes $U \times \mathbb{C}^N \to E^*|U$ a homeomorphism, where $\phi : E|U \to U \times \mathbb{C}^N$ is a local trivialization for E over the open subset $U \subseteq X$. Let $\psi : E|V \to V \times \mathbb{C}^N$ be another trivialization. Do ϕ and ψ define the same topology on $E^*|U \cap V$? Does the continuity of $(\psi \circ \phi^{-1})^* : (U \cap V) \times \mathbb{C}^N \to (U \cap V) \times \mathbb{C}^N$ follow from that of $(U \cap V) \times \mathbb{C}^N \to (U \cap V) \times \mathbb{C}^N$? For this, write the two chart changes in the form $U \cap V \to GL(N, \mathbb{C})$ and prove (!) that $GL(N, \mathbb{C}) \xrightarrow{*} GL(N, \mathbb{C})$ is continuous.]

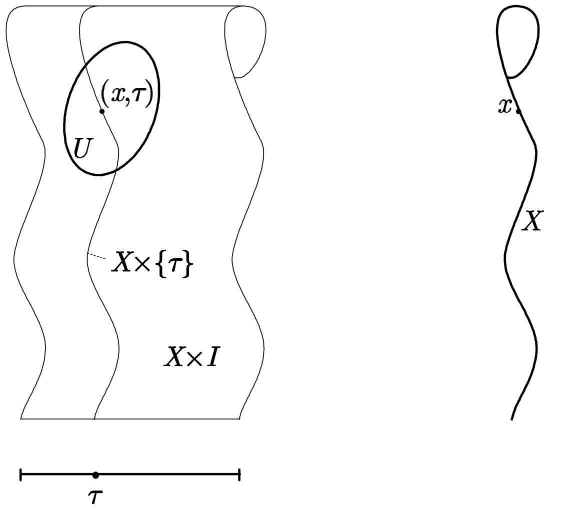
We denote the set of isomorphism classes of vector bundles over X by $\mathrm{Vect}(X)$, and let $\mathrm{Vect}_N(X)$ be the subset of $\mathrm{Vect}(X)$ consisting of the classes of bundles of dimension N. $\mathrm{Vect}(X)$ is an abelian semigroup under the operation $\oplus$. In $\mathrm{Vect}(X)$, there is a naturally distinguished element, namely the class of the trivial bundle of dimension N. A vector bundle over a point is a vector space, and hence $\mathrm{Vect}(X)$ can be identified with the semigroup $\mathbb{Z}^+$ of non-negative integers, in this case. However, in the general ease, when there are nontrivial bundles (see Exercise B.2 above), the isomorphism classes of vector bundles are not determined by their dimensions.

Two continuous mappings $f, h : X \to Y$ are **homotopic**, if there is a continuous map $F : X \times I \to Y$ $(I := [0, 1])$ such that $F_0 := F(\cdot, 0) = f$ and $F_1 := F(\cdot, 1) = h$. The map $f : X \to Y$ is a **homotopy-equivalence**, if there is a continuous map $g : Y \to X$ such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$ ("$\sim$" means "homotopic"). X and Y are then called **homotopy equivalent**. The set of homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$. X is called **contractible**, if X is homotopy equivalent to a point.

THEOREM B.5. *(i) If $f : X \to Y$ is a homotopy equivalence, then the transformation $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ of Exercise B.1d is bijective. (Assume X and Y compact.)*
(ii) If X is contractible, then every bundle over X is trivial, and $\mathrm{Vect}(X)$ is isomorphic to the non-negative integers.

PROOF. (ii) follows easily from (i), since $\mathrm{Vect}_N(P)$ consists only of the isomorphism class of the trivial bundle of dimension N, in the case where P is a point. (i) follows from the fact that $F_0^* E \cong F_1^* E$, if $F : X \times I \to Y$ is a homotopy and E is a vector bundle over Y. We give a proof of this in three steps:

Step 1: Let H be a vector bundle over $X \times I$ and $s \in C^0(H|X \times \{\tau\})$, $\tau \in I$. We show that s can be extended to a section $S \in C^0(H)$ with $S(\cdot, \tau) = s$. Since a section of a vector bundle can be regarded locally as a graph of a continuous vector-valued function, one can locally apply the Tietze Extension Theorem [**Du**]: We can find a neighborhood U about each (x, τ), and a section $t \in C^0(H|U)$ such that t and s coincide on $(X \times \{\tau\}) \cap U$.



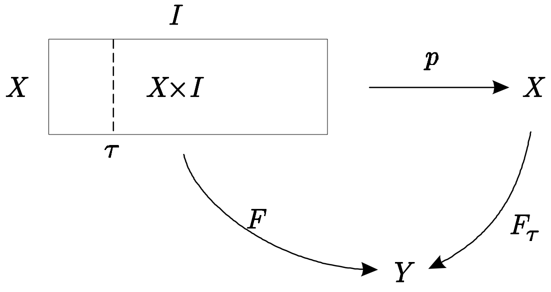
By the compactness of X, we can obtain a finite system $\{U_j\}$, $\{t_j\}$ with $X \times \{t\} \subset \cup U_j$. If $\{\phi_j\}$ is a C^0 partition of unity subordinate to $\{U_j\}$ (see also Theorem 6.4, p. 147), then we set

$$s_j := \begin{cases} t_j \phi_j & \text{on } U_j \\ 0 & \text{on } X \times I \setminus U_j. \end{cases}$$

By construction $s_j \in C^0(H)$; hence, $\sum s_j \in C^0(H)$ is a well-defined extension of S.
Step 2: From Step 1, we conclude that two vector bundles G and G' over $X \times I$ which are isomorphic over $X \times \{\tau\}$, are also isomorphic in a neighborhood of $X \times \{\tau\}$: Each $s \in \mathrm{Iso}(G|X \times \{\tau\}, G'|X \times \{\tau\})$ be regarded as a section of $H|X \times \{\tau\}$, where H is the bundle $L(G, G')$ of linear maps from fibers of G to corresponding fibers of G'. Let $S \in C^0(H)$ be an extension of s. Then the set

$U := \{x \in X \times I : S_z \in \operatorname{Hom}(G, G')$ is bijective$\}$ is open in $X \times I$ (by the classical zero determinant argument) and contains all of $X \times \{\tau\}$ by construction. Since the inverse map of $\mathrm{GL}(N, \mathbb{C})$ is continuous, it follows that the mapping $z \to S_z^{-1}$ is continuous, and hence a bundle isomorphism is defined on U.

Step 3. We now set $G := F^*E$ and $G' := p^*(F_\tau)^*E$, where $F_\tau(x) := F(x, \tau)$ and $p : X \times I \to X$ is the projection.



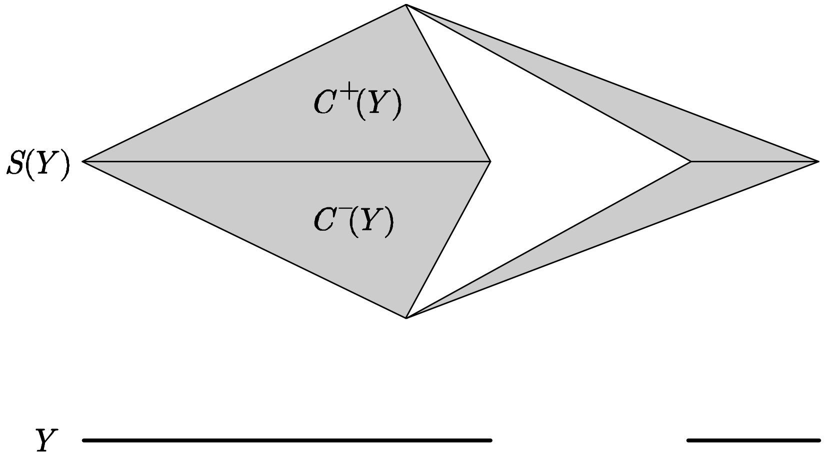
By Exercise B.1d), G and G' are isomorphic over $X \times \{\tau\}$), and by Step 2, they are also isomorphic in a whole neighborhood, which we can take to be a strip $X \times \delta(\tau)$, by the compactness of X. For all $\rho \in \delta(\tau)$, we then have $(F_\rho)^*E \cong (F_\tau)^*E$. Since the unit interval I is compact and connected, we obtain that the isomorphism classes of $(F_\tau)^*E$ do not depend on τ. □

Remark B.6. The statements proved in the first two steps of the proof, also apply to more general situations, and are occasionally formulated as independent theorems; e.g., see [**AB64b**, p.233f] or [**Ati67a**, p.16]. One can also express the result of our proof somewhat more generally (we write $X \times Z$ instead of $X \times I$): Each vector bundle E over the topological space $X \times Z$ (X and Z compact) can be regarded as a "continuous" family of vector bundles E over X where the parameter z is in Z, and the isomorphism classes of E in $\operatorname{Vect}(X)$ are locally constant.

Vector bundles are often given via a **clutching construction**: Let $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$, where all spaces are compact. Let E_i be a vector bundle over X_i and $\phi : E_1|A \to E_2|A$ an isomorphism. Then we define the vector bundle $E_1 \cup_\phi E_2$ over X as follows. As a topological space $E_1 \cup_\phi E_2$ is the quotient space of the disjoint union $E_1 + E_2$ under the equivalence relation which identifies $e_1 \in E_1|A$ with $\phi(e_1) \in E_2|A$. If we regard X as the corresponding quotient space of $X_1 + X_2$, then we obtain a natural projection $p : E_1 \cup_\phi E_2 \to X$ and $p^{-1}(x)$ has a natural vector space structure.

Exercise B.7. a) Show that $E_1 \cup_\phi E_2$ is a vector bundle.
b) Show that the isomorphism class of $E_1 \cup_\phi E_2$ depends solely on the homotopy class of the isomorphism $\phi : E_1|A \to E_2|A$.
[Hint for a): It remains only to show that $E_1 \cup_\phi E_2$ is locally trivial. Outside of A, this is clear. In order to extend a trivialization of E_1 on a neighborhood $V_1 \subseteq X$ of a point $a \in A$ to a trivialization of E_2 over V_2 with $a \in V_2 \subseteq X_2$, argue as in Step 2 of Theorem B.5. See also [**AB64b**, p.235] or [**Ati67a**, p.21]. For b): Reduce to Theorem B.5. See also the given sources.]

We now give $\mathrm{Vect}_N(X)$ a homotopy-theoretic interpretation, when X can be represented as the suspension $S(Y)$ of another space Y.[1] Here the **suspension** $S(Y)$ is the union of two cones over Y. Thus, we write $S(Y) : C^+(Y) \cup C^-(Y)$, where $C^+(Y) := Y \times [0, \frac{1}{2}]/Y \times \{0\}$ and $C^-(Y) := Y \times [\frac{1}{2}, 1]/Y \times \{1\}$. Then $Y = C^+(Y) \cap C^-(Y)$. We note that the suspension $S(S^n)$ of the n-sphere is homeomorphic to the $(n+1)$-sphere $S^{(n+1)}$.



Theorem B.8. *The clutching of trivial bundles over $C^+(Y)$ and $C^-(Y)$ defines a natural isomorphism* $[Y, \mathrm{GL}(N, \mathbb{C})] \overset{\cong}{\to} \mathrm{Vect}_N(S(Y))$.

Proof. (i) Each $l : Y \to GL(N, \mathbb{C})$ yields a bundle over SY via the clutching of the N-dimensional trivial bundles over the two cones, and homotopic maps l_0 and l_1 yield isomorphic bundles; see the proof of Theorem B.5(i). (ii) Conversely, we have the composition

$$\mathrm{Vect}_N(SY) \to \mathrm{Vect}_N(C^-Y) \oplus \mathrm{Vect}_N(C^+Y) \to [Y, \mathrm{GL}(N, \mathbb{C})],$$

the left arrow is given by the restrictions of the bundle, where one obtains trivial bundles since $C^\pm Y$ are contractible (see Theorem B.5(ii).) If $\alpha^\pm$ are such trivializations, then the right arrow is defined by taking the homotopy class of $(\alpha^+|Y)\,(\alpha^-|Y)^{-1} : Y \to \mathrm{GL}(N, \mathbb{C})$, which actually on depends on the homotopy classes of $\alpha^\pm$ and hence only on the isomorphis class in $\mathrm{Vect}_N(SY)$. (iii) By construction the functions given in (i) and (ii) are inverse to each other. □

Exercise B.9. Show $H_{\mathbb{C}^2} \cong \mathbb{C}_{B_0} \cup_a \mathbb{C}_{B_\infty}$, where $H_{\mathbb{C}^2}$ is the complex line bundle over $\mathbb{PC}^2 = \mathbb{C} \cup \{\infty\} = S^2 = B_0 \cup B_\infty$ defined in Exercise B.2a; here (z_0, z_1) are homogeneous coordinates for $\mathbb{PC}^2$ with $(0, 1) = \infty$, and $z = z_1/z_0$ is the coordinate for $\mathbb{C}$, $B_0 := \{z \in \mathbb{C} : |z| \le 1\}$, and $B_\infty := \{z \in \mathbb{C} : |z| \ge 1\} \cup \{\infty\}$ (the two canonical hemispheres of S^2). Finally, $a : z \mapsto z$, is the standard map $S^1 \to \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$.

Exercise B.10. Show that for each bundle E, there is a bundle F such $E \oplus F$ is trivial. [Hint: Show first with the help of a finite open cover of the compact parameter space X and a suitable partition of unity, that $C^0(E)$ contains an "ample"

---

[1] With the concept of Grassmann manifolds, one can give a homotopy-theoretic definition of $\mathrm{Vect}_N(X)$ for *arbitrary* X; see [**Ati67a**, p.24-30].

subspace; i.e., a subspace $V \subseteq C^0(E)$ such that each point of E is in the image of a section $s \in V$. If $\dim V = N$, then we have an epimorphism $\phi : X \times \mathbb{C}^N \to E$, and consequently there is an isomorphism $E \oplus F \to X \times \mathbb{C}^N$ where F is the kernel bundle of ϕ; see also [**Ati67a**, p.26 f].]

EXERCISE B.11. Let X be a topological space which possesses in addition the structure of a C^∞ manifold of dimension n (see Chapter 6).
a) Show that the tangent bundle TX is a (real) vector bundle over X; do the same for the "normal bundle" NX, when X is a submanifold of a Riemannian manifold Y.
b) When may one call a continuous vector bundle over X, whose total space is a C^∞ manifold, a C^∞ vector bundle? Show that each (continuous) vector bundle over X is isomorphic to a C^∞ vector bundle.
[Hint for a): First investigate the case $X = S^1$ and show that TS^1 is isomorphic to the real line bundle defined in Exercise B.1b. In the general case, this "direct" method is also possible, since one can (see Theorem 6.7c, p. 150) embed X into a high dimensional Euclidean space, and thus realize TX as a (real) subbundle of a higher dimensional trivial bundle. It is easier (especially since TX is in general not trivial; e.g., for $X = S^2$, see Exercise 13.23, p. 241) to carry out a local analysis, where each chart u from the C^∞ atlas yields a local trivialization of TX via the differential forms $(du_1, ..., du_n)$; see Chapter 6. See also the discussion below.
For b): For the definition of C^∞ vector bundles, see also [**BJ**, Ch.3], and for the topological equivalence of the categories of C^0 and C^∞ vector bundles, see the Whitney Approximation Theorem in [**BJ**, p.66]. Details of the argument are in [**Hir**, p.101], where it is shown that one can make E itself into a C^∞ vector bundle.]

REMARK B.12. From Exercise B.11b, it follows (without loss of generality) that we only need to investigate C^∞ vector bundles, if the base is C^∞. Exercise B.10 does not say that we always encounter trivial bundles (compare with the analogous - but deeper embedding theorem for manifolds, Theorem 6.7c, p. 150): Roughly, the carrying along of additional "irrelevant" parameters is not only redundant, but can also produce so much "noise" that this noise destroys the structure of the problem or renders it unrecognizable. This is the case with the index problem for elliptic operators, whose solution consists exactly in distinguishing certain vector bundles generated by the symbol of the operator; see Part 3.

It is possible to construct (see Exercise B.13 below), from the tangent bundles, bundles of *exterior differential forms* by means of multilinear algebra. These are an important tool in describing physical laws mainly in the areas of electromagnetism and special relativity. This is the case when empirical relationships are to be expressed in terms of an integral in such a way that the physicist or engineer can pursue qualitative and quantitative changes resulting from modifications of the integrand or the domain of integration. For this reason, exterior bundles are of interest also in differential topology; see Chapter 16.

We briefly summarize (details are found in [**Brd**, p.260f], [**KN63**, p.17f], and [**UN**, p.111-161] and the literature given there): For a real n-dimensional vector space V, we form the vector space $\Lambda^p(V)$ of p-fold **skew-symmetric tensors** (or p-**vectors**); these are the multilinear maps

$$V^* \times \overset{p \text{ times}}{\cdots} \times V^* \to \mathbb{R},\ p \in \mathbb{N},\ V^* := L(V, \mathbb{R})$$

which change, under a permutation of the arguments, by a factor equal to the sign of the permutation. One sets $\Lambda^0(V) := \mathbb{R}$ and obtains $\Lambda^1(V) = V$, $\Lambda^{n-1}(V) \cong V$, $\Lambda^n(V) \cong \mathbb{R}$ and $\Lambda^p(V) = \{0\}$ for $p > n$. For $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$, we define $v \wedge w \in \Lambda^{p+q}(V)$ by

$$(v \wedge w)(a_1, ..., a_{p+q}) := \frac{1}{p!q!} \sum\nolimits_\sigma \operatorname{sgn}(\sigma)\,(v \otimes w)\,(a_{\sigma(1)}, ..., a_{\sigma(p+q)})$$

(sum over all permutations), which gives the **exterior multiplication** $\Lambda^p(V) \times \Lambda^p(V) \to \Lambda^{p+q}(V)$. This multiplication makes $\Lambda^*(V) := \sum_{p=0}^n \Lambda^p(V)$ a graded algebra, the **exterior algebra** of V.

If $e_1, ..., e_n$ is a basis of V, then the $\binom{n}{p}$ forms $e_{i_1} \wedge \cdots \wedge e_{i_p}$ with $1 \le i_1 < \cdots < i_p \le n$ yield a basis for $\Lambda^p(V)$. With this property, $\Lambda^p(V)$ is occasionally defined (in order to avoid the suggestive but tedious definition via maps) as the space of ***p*-vectors**: The space formal linear combinations of the p-tuples of basis vectors $e_{i_1} \wedge \cdots \wedge e_{i_p}$ with only the relation $e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_p)} = \operatorname{sgn}(\sigma) e_{i_1} \wedge \cdots \wedge e_{i_p}$.

A scalar product (= inner product) for V induces a scalar product $\langle \cdot, \cdot \rangle$ for $\Lambda^p(V)$, and declaring an orthonormal basis $e_1, ..., e_n$ of V to be positively, oriented yields an explicit isomorphism $\Lambda^n(V) \cong \mathbb{R}$ via $e_1 \wedge \cdots \wedge e_n \to 1$, which only depends on the chosen orientation and scalar product. The (Hodge) *star operator*

$$* : \Lambda^p(V) \to \Lambda^{n-p}(V) \text{ is given via } u \wedge *v = \langle u, v \rangle\, e_1 \wedge \cdots \wedge e_n \text{ for all } u \in \Lambda^p(V).$$

EXERCISE B.13. Let X be a closed C^∞ manifold of dimension n with cotangent bundle T^*X.
a) Show that the family of vector spaces $\Lambda^p(T_x^*X)$, $x \in X$, yields a real vector bundle of fiber dimension $\binom{n}{p}$ over X in a natural way. We denote this bundle by $\Lambda^p(T^*X)$.
b) Customarily, one writes $\Omega^p(X) := C^\infty(\Lambda^p(T^*X))$, which is the space of **exterior differential *p*-forms** on X. For a C^∞ function f, consider the differential df (see also Section 6.2) and show that the operator $d : \Omega^0(X) \to \Omega^1(X)$ uniquely extends to a linear differential operator $d : \Omega^p(X) \to \Omega^{p+1}(X)$ of first order (see Chapter 6) for each p, such that $d^2 := d \circ d = 0$ and

$$d\,(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p\, \alpha \wedge d\beta$$

for all $\alpha \in \Omega^p(X)$ and $\beta \in \Omega^q(X)$.
c) Show that for an oriented Riemannian manifold X, we have the "Hodge duality" $* : \Omega^p(X) \cong \Omega^{n-p}(X)$.
d) Prove that for a compact, oriented, n-dimensional, Riemannian manifold X with boundary ∂X, we have Stokes' Theorem

$$\int_X d\omega = \int_{\partial X} \omega, \text{ for } \omega \in \Omega^p(X).$$

[Hint for a): In principle, use the same mechanism as in Exercise B.4. Note that for charts u and w for X in a neighborhood of $x \in X$, we have the simple transformation rules, e.g., for a 1-form $v \in \Omega^1(X)$,

$$v(x) = \sum\nolimits_{j=1}^n a_j(x)\, du^j\big|_x = \sum\nolimits_{i=1}^n b_i(x)\, dw^i\big|_x, \text{ where}$$

$$b_i(x) := \sum\nolimits_{j=1}^n a_j(x) \frac{\partial \left(u \circ w^{-1}\right)^j}{\partial x^i} \left(w\left(x\right)\right).$$

For b): d is characterized by the Leibniz rule

$$d(v \wedge w) = dv \wedge w + (-1)^p v \wedge dw, \text{ for } v \in \Omega^p(X),\ w \in \Omega^q(X).$$

How is d written in local coordinates?
For d): See [**GP**, p.182-187]. Incidentally, here one really needs the orientation.]

The operator $d : \Omega^p(X) \to \Omega^{p+1}(X)$ is known as the *exterior derivative operator* or *exterior differentiation*. In local coordinates, $x^1, \ldots, x^n$ defined on a coordinate neighborhood $U \subseteq M$, a form $\alpha \in \Omega^p(X)$ can be written as

$$\frac{1}{p!} \sum_{i_1,\ldots,i_p=1}^{n} \alpha_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where the $\alpha_{i_1\cdots i_p} \in C^\infty(U)$ are antisymmetric in the indices $i_1, \ldots, i_p$. On U,

$$\begin{aligned} d\alpha &= \frac{1}{p!} \sum_{i_1,\ldots,i_p=1}^{n} d\left(\alpha_{i_1\cdots i_p}\right) dx^{i_1} \wedge \cdots \wedge dx^{i_1} \\ &= \frac{1}{p!} \sum_{i,i_1,\ldots,i_p=1}^{n} \tfrac{\partial}{\partial x^i}\left(\alpha_{i_1\cdots i_p}\right) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_1} \end{aligned}$$

However, since $d : \Omega^p(X) \to \Omega^{p+1}(X)$ is uniquely determined by the coordinate-free operation $d : \Omega^0(X) \to \Omega^1(X)$, one should be able to express d in a coordinate-free manner. For this purpose (and because it is an important and basic notion), we introduce Lie differentiation. Let $A \in C^\infty(TX)$ be a vector field on X. The theory of systems of first-order ordinary differential equations guarantees that for each point $p \in X$, there is $\varepsilon > 0$ and a curve $\alpha_p : (-\varepsilon, \varepsilon) \to X$ such that $\alpha_p'(t) = A_{\alpha_p(t)}$. This curve α is known as the integral curve of A through p. Furthermore, ε can be chosen so that for all q in some neighborhood U of p, the integral curve $\alpha_q : (-\varepsilon, \varepsilon) \to X$ exists, and there is a there is a well-defined C^∞ map

$$\alpha : U \times (-\varepsilon, \varepsilon) \to X \text{ given by } \alpha(q,t) := \alpha_q(t),$$

such that $\alpha_t := \alpha(\cdot, t) : U \to \alpha(U,t)$ is a diffeomorphism for each $t \in (-\varepsilon, \varepsilon)$. Moreover, $\alpha_{t+s} = \alpha_t \circ \alpha_s$ whenever both sides are defined. In particular $\alpha_t^{-1} = \alpha_{-t}$ on $\alpha(U,t) \cap U$ which is nonvoid for small t. Given a second vector field $B \in C^\infty(TX)$, we have $\left(\alpha_t^{-1}\right)_*\left(B_{\alpha_t(p)}\right) \in T_pX$ and so the curve

$$t \mapsto \left(\alpha_t^{-1}\right)_*\left(B_{\alpha_t(p)}\right) = (\alpha_{-t})_*\left(B_{\alpha_t(p)}\right)$$

lies in the *single* vector space T_pX. It then makes sense to differentiate this curve at $t = 0$ to obtain a vector in T_pX which is known as the **Lie derivative** of B with respect to A at p, namely

$$\left(L_A B\right)_p := \tfrac{d}{dt}\left(\alpha_{-t}\right)_*\left(B_{\alpha_t(p)}\right)\Big|_{t=0}$$

The assignment $p \mapsto \left(L_A B\right)_p$ defines a vector field $L_A B \in C^\infty(TX)$. If A and B are vector fields on $\mathbb{R}^n$ and p, $\delta p \in \mathbb{R}^n$, then (where $\approx$ denotes equality modulo terms of first-order in t) we have

$$\left(\alpha_{-t}\right)_*(\delta p) \approx \delta p - t\,(dA)_p(\delta p) \text{ and } B_{\alpha_t(p)} \approx B_{p+tA_p} \approx B_p + t\,(dB)_p(A_p).$$

Thus,

$$\begin{aligned}&\left(\alpha_{-t}\right)_*\left(B_{\alpha_t(p)}\right) \approx B_{\alpha_t(p)} - t\left(dA\right)_p\left(B_{\alpha_t(p)}\right) \approx B_p + t\left(dB\right)_p\left(A_p\right) - t\left(dA\right)_p\left(B_p\right)\\ \Rightarrow\quad &\left(L_A B\right)_p = \left(dB\right)_p\left(A_p\right) - \left(dA\right)_p\left(B_p\right) = A_p\left[B\right] - B_p\left[A\right].\end{aligned}$$

Hence, $L_A B = -L_B A$ and viewing the vector fields A and B differential operators on functions f, we have

$$\left(L_A B\right)_p(f) = \left(df\right)_p\left(A_p\left[B\right] - B_p\left[A\right]\right) = A_p\left[B\left[f\right]\right] - B_p\left[A\left[f\right]\right],$$

and so as a differential operator $L_A B$ is the commutator of A and B, i.e.,

$$L_A B = A \circ B - B \circ A = [A, B]$$

In local coordinates $x^1, \ldots, x^n$, we have (automatically summing over repeated indices)

$$\begin{aligned}L_A B &= L_A B\left[x^i\right]\partial_{x^i} = \left(A\left[B\left[x^i\right]\right] - B\left[A\left[x^i\right]\right]\right)\partial_{x^i}\\ &= \left(A^j \partial_{x^j} B^i - B^j \partial_{x^j} A^i\right)\partial_{x^i}.\end{aligned}$$

For a one-form ω, we then have the coordinate-free relation

$$d\omega\left(A, B\right) = A\left[\omega\left(B\right)\right] - B\left[\omega\left(A\right)\right] - \omega\left(\left[A, B\right]\right), \text{ since} \tag{0.1}$$

$$\begin{aligned}& A\left[\omega\left(B\right)\right] - B\left[\omega\left(A\right)\right] - \omega\left(\left[A, B\right]\right)\\ =\ & A\left[\omega_j B^j\right] - B\left[\omega_j A^j\right] - \omega\left(A^i \partial_{x^i}\left(B^j\right)\partial_{x^j} - B^i \partial_{x^i}\left(A^j\right)\partial_{x^j}\right)\\ =\ & A^i\left(B^j \partial_{x^i}\omega_j + \omega_j \partial_{x^i} B^j\right) - B^i\left(A^j \partial_{x^i}\omega_j + \omega_j \partial_{x^i} A^j\right)\\ & -\omega_j A^i \partial_{x^i} B^j + \omega_j B^i \partial_{x^i} A^j\\ =\ & \partial_{x^i}\omega_j\left(A^i B^j - B^i A^j\right) = \partial_{x^i}\left(\omega_j\right)\left(dx^i \wedge dx^j\right)\left(A, B\right) = d\omega\left(A, B\right).\end{aligned}$$

EXERCISE B.14. For $\psi \in \Omega^2(X)$ and vector fields $A, B, C \in C^\infty(TX)$, show that

$$\begin{aligned}d\psi\left(A, B, C\right) &= A\left[\psi\left(B, C\right)\right] + B\left[\psi\left(C, A\right)\right] + C\left[\psi\left(A, B\right)\right]\\ &\quad -\Omega\left(\left[A, B\right], C\right) - \Omega\left(\left[C, A\right], B\right) - \Omega\left(\left[B, C\right], A\right)\\ &= \mathfrak{S}\left(A\left[\psi\left(B, C\right)\right] - \Omega\left(\left[A, B\right], C\right)\right),\end{aligned}$$

where $\mathfrak{S}$ denotes the sum over cyclic permutations of (A, B, C). [Hint. Consider the case $\psi = \omega \wedge \varphi$, for $\omega, \varphi \in \Omega^1(X)$, and use $d\psi = d\omega \wedge \varphi - d\varphi \wedge \omega$. The general case follows by linearity, since any 2-form is locally a sum of wedges of 1-forms.]

More generally, one can show by induction that for $\psi \in \Omega^k(X)$

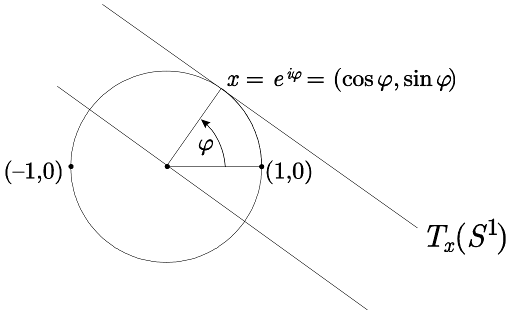
$$\begin{aligned}d\psi\left(A_1, \ldots, A_{k+1}\right) &= \sum_{i=1}^{k+1}\left(-1\right)^{i+1} A_i\left[\psi\left(A_1, \ldots, \widehat{A_i}, \ldots, A_{k+1}\right)\right]\\ &\quad + \sum_{1 \leq i < j \leq k+1}\left(-1\right)^{i+j}\psi\left(\left[A_i, A_j\right], A_1, \ldots, \widehat{A_i}, \ldots, \widehat{A_j}, \ldots, A_{k+1}\right),\end{aligned} \tag{0.2}$$

where $\widehat{A_i}$ indicates that A_i is omitted. For a proof, see [**KN63**, p.36]. Note that the extra numerical factor of $1/\left(k+1\right)$ in the formula of [**KN63**, p.36] is ultimately due to their convention that $\left(dx^1 \wedge \cdots \wedge dx^n\right)\left(\partial_{x^1}, \ldots, \partial_{x^n}\right) = 1/n!$ (see [**KN63**, p.7]), while our convention is that $\left(dx^1 \wedge \cdots \wedge dx^n\right)\left(\partial_{x^1}, \ldots, \partial_{x^n}\right) = 1$.

The idea of a vector bundle originates in the analysis of non-Euclidean manifolds. Otherwise, according to Theorem B.5(ii), there are only trivial bundles, since Euclidean space is contractible. As an example, let $c : I \to X$ be a differentiable path in a manifold X. Classical mechanics considers the velocity vector $c'(t)$ for $t \in I$. For the physicist it was always clear (by reasons of physics) how $c(t)$ is multiplied by a scalar, how $c_1'(t_1)$ and $c_2'(t_2)$ are added or how the equality of $c_1'(t_1)$ and $c_2'(t_2)$ is checked when $c_1, c_2 : I \to X$. are two paths in X with $c_1(t_1) = c_2(t_2)$. From the point of view of physics no confusion between a velocity vector and a position vector was conceivable, but the earliest mathematical abstractions could not express the difference: If the position vectors $c(t)$ are represented as triples of real numbers $(c_1(t), c_2(t), c_3(t))$, then the velocity vector is $(c_1'(t), c_2'(t), c_3'(t))$ where $c_i'(t)$ is the derivative of c_i' at t. Thus, $c(t)$ and $c'(t)$ are both elements of the single vector space $\mathbb{R}^3$. This low level of abstraction was fully sufficient as long as X was Euclidean space (actually an affine space – but choose a base point and call it 0). In this case there is indeed a natural interpretation of the velocity vector $c'(t)$ at the space point $c(t)$ as a velocity vector $\widetilde{c}'(t)$ at the point $\widetilde{c}(t) = 0$. In fact, consider the translated path $\widetilde{c} : I \to X$ with $\widetilde{c}(\tau) := c(\tau) - c(t)$, $\tau \in I$. This means that the tangent spaces at the various points of X can be identified canonically (via the retraction $r : X \to \{0\}$) with the tangent space at the point 0. In the language of vector bundles, we could say $TX \cong r^*(TX|\{0\})$. Here TX is the, totality of all velocity vectors, the fiber T_xX is $\mathbb{R}^3$, the restricted bundle $TX|\{0\}$ is precisely the space $\{0\} \times \mathbb{R}^3 \cong \mathbb{R}^3$ and the induced bundle $r^*(\{0\} \times \mathbb{R}^3)$ is the trivial bundle $X \times \mathbb{R}^3$.

Let us next consider the case that X is a submanifold of a Euclidean space Y, say the 2-sphere in 3-space. Every velocity vector $c'(t)$ can be considered an element of the tangent space of Y at the point $c(t)$ and hence an element of the tangent space of Y at 0 (due to the translation described above), i.e., as n-tuple of derivatives of the coordinate functions of c. Already by reason of dimensions it is clear that, in general, the tangent space T_xX of all velocity vectors X of paths through x cannot be identified with the full tangent space T_0Y but only with some subspace and it may be different for each x

EXAMPLE B.15. $X = S^1$ and $Y = \mathbb{R}^2$



If we follow the translation with a rotation through the angle ϕ, then we have identified all tangent spaces TS^1 with the same subspace of T_0S^2 or with the tangent space $T_{(0,1)}S^1$. We write $TS^1 = q^*(TS^1)/\{(1,0)\}$, where $q : S^1 \to (1,0)$ is the retraction and $TS^1|\{(1,0)\}$ can be identified with $\{(1,0)\} \times \mathbb{R}$. Consequently $TS^1 \cong S^1 \times \mathbb{R}$, i.e., the tangent bundle of S^1 is trivial.

In analytic terms this circumstance is expressed by saying that there exists a nowhere vanishing tangential vector field on S^1; i.e., at each point a non-vanishing velocity vector can be chosen and in a continuous fashion. For example choose at $x = (\cos\theta, \sin\theta)$ the unit velocity vector $c'(\phi/2\pi)$, where $c : I \to X$ is given by $c(t) = (\cos 2\pi t, \sin 2\pi t)$. Usually we write $\frac{d}{d\theta}\big|_x$ instead of $c'(\phi/2\pi)$. Then $\frac{d}{d\theta}$ is the nowhere vanishing vector field and it defines an isomorphism $TS^1 \cong S^1 \times \mathbb{R}$. The isomorphism is, for all $x \in S^1$, given by the map $T_xS^1 \to \mathbb{R}$ which assigns to a velocity vector the value λ, if it is equal to $\lambda \cdot \frac{d}{d\theta}\big|_x$.

EXAMPLE B.16. $X = S^2$ and $Y = \mathbb{R}^3$. Here the situation is different. There is no canonical way of identifying all tangent spaces. In other words, the tangent bundle TS^2 is non-trivial. If not there would exist two tangential vector fields on S^2 which are linearly independent at each point of S^2. But on S^2 there is not even a single vector field that vanishes nowhere (Exercise 13.23, p. 241). This example shows that it might be useful (indeed necessary) to distinguish tangent spaces at different points. As one goes on to consider manifolds without an explicitly given imbedding in Euclidean spaces, as in physics with the theory of relativity, the notion of a bundle becomes indispensable.

The modern concept of a bundle evolved from the topology and geometry of manifolds as practiced by Heinz Hopf and others since the 1920's. In the 1950's the notion was precisely formulated, and the classification of bundles and their systematic employment in deep problems of geometry and analysis started. (For the theory of "characteristic classes", see e.g.,[**Hir66a**, p.49 f], [**KN69**, Chapter XII], and Section 18.7.) Crudely expressed, the success of these methods derives from their utilizing given or manufactured "classical" structures (such as the tangent bundles or bundles of differential forms) on the manifolds under discussion to the greatest extent possible and thus shifting the plane of study from manifolds – which are conceptually simpler but harder to understand – to vector bundles which are more easily analyzed:

While for topological manifolds and other "triangulable" spaces, one depends at first on the combinatorial methods of the analysis of "cell decompositions", and while for differentiable manifolds only the group diffeomorphisms is *a priori* available for investigation, vector bundles offer more opportunity for manipulations because of their richer structure. Principally, large parts of linear algebra can be used directly (Incidentally, these play an important role also for the other method, albeit under the surface.) For example (see above Exercise B.4) one can perform linear constructions with vector bundles such as forming direct sums and quotients, which is impossible to do with manifolds. Also, "clutching functions" which can be used to build complicated manifolds from simpler ones (see e.g., Exercise 6.20, p. 158), i.e., the diffeomorphisms, become linear only in the first derivative ("functional matrix" or "Jacobian"). In contrast, Theorem B.5 shows how much (via linear clutching functions) the topology of vector bundles can be reduced to the geometry of the matrix spaces of linear algebra.

At the start of the 1960's linear algebra had "matured" enough with the Periodicity Theorem (for the (stable) homotopy groups of invertible matrices) discovered by Raoul Bott just before. Michael Francis Atiyah and Friedrich Hirzebruch extracted from these methods an abstract formalism – K-theory – which they developed as a generalized cohomology theory, using stability classes of vector bundles; see Chapter 13.